

Asymptotic Conditional Probability in Modal Logic: A Probabilistic Reconstruction of Nonmonotonic Logic

Riccardo Rosati

Dipartimento di Informatica e Sistemistica
Università di Roma “La Sapienza”
rosati@dis.uniroma1.it

Georg Gottlob

Institut für Informationssysteme
Technische Universität Wien
gottlob@dbai.tuwien.ac.at

*Dedicated to Victor Marek
on his 65th birthday*

Abstract

We analyze the asymptotic conditional validity of modal formulas, i.e., the probability that a formula ψ is valid in the finite Kripke structures in which a given modal formula φ is valid, when the size of these Kripke structures grows to infinity. We characterize the formulas ψ that are almost surely valid (i.e., with probability 1) in case φ is a flat, S5-consistent formula, and show that these formulas ψ are exactly those which follow from φ according to the nonmonotonic modal logic S5_G. Our results provide – for the first time – a probabilistic semantics to a well-known nonmonotonic modal logic, establishing a new bridge between nonmonotonic and probabilistic reasoning, and give a computational account of the asymptotic conditional validity problem in Kripke structures.

1 Introduction

Asymptotic Probabilities in Modal Logic. Asymptotic (or limit) probabilities of classical logic formulas have been investigated in various contexts (Glebskii *et al.* 1969; Fagin 1976; Compton 1988; Kolaitis and Vardi 1990; Bars 1998). Halpern and Kapron have analyzed asymptotic probability in modal logic (Halpern and Kapron 1994), where, instead of relational structures, Kripke structures are considered, and where the size of a structure is measured in terms of the number of its worlds. Among various results,¹ they have shown that every modal formula is either almost surely true or almost surely false in finite Kripke structures. Thus, there is a *0-1 law* for modal logic K, analogous to the 0-1 law for function-free first-order logic (Fagin 1976; Glebskii *et al.* 1969).

To give a simple example, if p is a propositional letter of a considered finite alphabet \mathcal{A} , then the modal formula² Mp is almost surely true (w.r.t. modal logic K). In fact, it can be seen that if $n \rightarrow \infty$, then the cardinality of the set

A_n of Kripke structures for \mathcal{A} of size $\leq n$ satisfying Mp (i.e., where at least one world satisfying p is reachable from each world) grows much faster than the cardinality of the set B_n of Kripke structures which do not satisfy Mp , and thus $\lim_{n \rightarrow \infty} |A_n|/|C_n| = 1$ where C_n is the set of *all* Kripke structures with n worlds over the considered alphabet.

Assigning asymptotic probabilities to modal formulas provides an interesting nonstandard semantics to modal logics and has important connections to philosophy and to artificial intelligence:

PHILOSOPHY: It has been observed (Halpern and Kapron 1994; Gottlob 1999) that the modal formulas that are almost surely true in all Kripke models are exactly those formulas which are valid in Carnap’s modal logic, exposed in his well-known foundational treatise *Meaning and Necessity* (Carnap 1947). Carnap argued that precisely these formulas are those to be considered logically true (*L-true*). The same logic has since be considered by various philosophers and logicians (Gottlob 1999).

ARTIFICIAL INTELLIGENCE: Various *nonmonotonic modal logics* have been defined in the literature. Examples are autoepistemic logic (Moore 1985), nonmonotonic logics K, S4, etc. (Marek and Truszczyński 1993), the logic MBNF (Lifschitz 1994) and the logic of minimal knowledge S5_G (Halpern and Moses 1985). In these logics, the modal operator K is interpreted as an epistemic operator of knowledge or of belief. As observed in (Gottlob 1999), the formulas that can be derived from an empty set of premises in *all these logics* precisely coincide with those formulas which are almost surely true in all Kripke structures. This set of formulas is furthermore identical to the *stable set* (Stalnaker 1993) based on the empty set of formulas, thus, in a sense, to the “absolute” stable set, containing only those formulas that are epistemic consequences of the empty theory, i.e., that can be assumed in the case of total factual ignorance (for a definition of stable sets, see Section 4). We thus retain that in nonmonotonic and epistemic logics, in absence of further knowledge, a modal formula is considered true if and only if this formula is almost surely valid.

Reasoning and Asymptotic Conditional Truth. While almost sure validity provides an appealing probabilistic semantics of truth in the case of total factual ignorance, this does not yet allow us to *reason* on the basis of *premises*,

Copyright © 2007, authors listed above. All rights reserved.

¹Some of the results of (Halpern and Kapron 1994) have been corrected in (Bars 2002), but those referred-to in the present paper are perfectly correct in (Halpern and Kapron 1994).

²We here use K as the symbol for the necessity (= knowledge) operator and M ($\equiv \neg K \neg$) for the possibility operator.

which is the the most important goal of all logical formalisms, and, in particular, of nonmonotonic and epistemic logics. In the setting of limit probabilities, the inference $\varphi \models \psi$ of a (possibly modal) formula ψ from a knowledge base (theory) φ would most intuitively correspond to the statement that the conditional probability $P(\psi|\varphi)$ is asymptotically equal to 1. Assuming a uniform probability distribution of Kripke structures, this means that for the sets A_n and B_n of Kripke models of size n satisfying $\psi \wedge \varphi$ and φ , respectively, the limit $\lim_{n \rightarrow \infty} |A_n|/|B_n|$ exists and is equal to 1.

There is, in general, no 0-1 law for conditional probabilities in our setting, which immediately follows from previous results (see e.g. (Grove *et al.* 1996b)) and from the fact that modal logic corresponds to a fragment of function-free first-order logic. The existence and determination of conditional probabilities in first and higher order logics has been the subject of several studies (Grove *et al.* 1996b; 1996a). However, to our best knowledge, conditional probabilities for modal logics have never been studied.

Observe that reasoning via conditional limit probabilities as explained above clearly constitutes a form of nonmonotonic reasoning. For example, if p is a propositional letter, then $\models_{\text{lim}} Mp$ but $K\neg p \not\models_{\text{lim}} Mp$, thus, adding a premise may invalidate a consequence (here \models_{lim} denotes inference of almost sure formulas under conditional limit probabilities). The nonmonotonic behaviour of conditional inference has been pointed out in the context of FO and higher order classical logic (see e.g. (Grove *et al.* 1996b)). We deem the context of modal logic particularly interesting, because most nonmonotonic logics that have been defined are modal logics. It would thus be very interesting to know how the nonmonotonic modal logic obtained from conditional limit probabilities of Kripke structures relates to well-known and well-studied nonmonotonic modal logics.

Nonmonotonic Modal Logics. For every “classical” monotonic system of modal logic S , a nonmonotonic version NM- S is obtained by the following definition: A set E of modal formulas is an NM- S -expansion of a knowledge base φ iff $E = Cn_S(\varphi \cup \{\neg K\gamma \mid \gamma \in \mathcal{L}_K(\mathcal{A}) - E\})$, where Cn_S is the consequence operator according to modal logic S and $\mathcal{L}_K(\mathcal{A})$ is the underlying modal language, i.e., the set of all possible modal formulas over the alphabet of propositions \mathcal{A} ³. In particular, Moore’s autoepistemic logic corresponds to NM-KD45. For more background, consult (Marek and Truszczyński 1993).

It was often criticized that the above fixed-point equation is somewhat too liberal, because it allows a theory to have expansions that are not sufficiently “grounded” in the premises and contain positive knowledge that a rational agent should never conclude from the premises. For this reason, *ground nonmonotonic modal logics* have been defined by restricting the introspection of the agent to non-modal sentences. The notion of groundedness has a rather intuitive

motivation: in fact, it corresponds to discarding any reasoning based on epistemic assumptions, which, for example, would enable the agent to conclude that something is true in the world, by assuming to know it. The fixed-point equation defining a S_G -expansion of φ is: $E = Cn_S(\varphi \cup \{\neg K\gamma \mid \gamma \in \mathcal{L}(\mathcal{A}) - E\})$, where $\mathcal{L}(\mathcal{A})$ denotes the set of all objective (i.e., non-modal) formulas of the underlying propositional language. According to this equation, we can associate a logic S_G to each modal logic S . Given $\varphi \in \mathcal{L}_K(\mathcal{A})$, $\psi \in \mathcal{L}_K(\mathcal{A})$, we say that ψ is entailed by φ in S_G (and write $\varphi \models_{S_G} \psi$) iff ψ belongs to all S_G -expansions for φ .

Among all ground nonmonotonic logics, the logic $S5_G$ (which, unlike NM-S5, is a true nonmonotonic logic) has received considerable attention in the literature and is generally referred to as the *logic of minimal knowledge* (Halpern and Moses 1985) (or the *logic of maximal ignorance*). In fact, independently from its fixed-point characterization, $S5_G$ was characterized on a semantic basis, by means of a preference criterion among the models of an agent’s knowledge selecting just those models in which objective knowledge (i.e., the set of formulas of type $K\varphi$ such that $\varphi \in \mathcal{L}(\mathcal{A})$) is minimal (Halpern and Moses 1985; Shoham 1987; Lifschitz 1994). As shown in (Shoham 1987), $S5_G$ has a simple, elegant model theory: The $S5_G$ models of a theory T are precisely those Kripke structures which are universal (i.e., totally connected) and are maximal set of worlds (w.r.t. set-containment). For details on ground nonmonotonic modal logics and $S5_G$, see (Halpern and Moses 1985; Lifschitz 1994; Donini *et al.* 1997).

Main Problems Studied. The investigations reported in this paper were motivated by the following questions:

1. Given a formula φ , which formulas ψ are almost surely true in the Kripke models of φ , i.e., how can we characterize the formulas ψ that are true with limit probability one in the Kripke models of φ , when the size of these Kripke models grows towards infinity?
2. Given that modal inference under almost sure validity constitutes a form of nonmonotonic reasoning, to which of the nonmonotonic modal logics from the above-mentioned plethora of logics does this form of reasoning best correspond?
3. Can we characterize the set of formulas φ that guarantee a 0-1 law for the asymptotic probability $P(\psi|\varphi)$ for arbitrary modal formulas ψ ?
4. What is the complexity of modal reasoning based on asymptotic conditional probabilities?

For studying these questions, we make a very weak assumption on φ , which assures that φ does not contradict some principles of knowledge: we assume that φ is $S5$ -consistent, i.e., we assume that φ is consistent with the axioms of (monotonic) $S5$, which means that φ admits at least one $S5$ -model. This assumption is indeed very weak. It does neither mean that φ contains the $S5$ axioms, or that these axioms should follow from φ , nor that φ has to be interpreted under $S5$ Kripke structures only. Our assumption merely requires that formulas such as $Kp \wedge K\neg p$ which bluntly contradict some axioms of (monotonic) $S5$ cannot

³This definition is applicable to a large range of modal logics, but a collapse happens at $S5$ because NM- $S5=S5$ (Marek and Truszczyński 1993).

be deduced from φ . If φ satisfies this requirement, we say that φ is *knowledge-consistent*. We make this assumption because knowledge-inconsistent theories are well-known to be *inconsistent* for all currently known nonmonotonic modal logics anyway, i.e., in each such nonmonotonic logic, a knowledge-inconsistent formula φ entails a contradiction and thus all formulas of the modal language $\mathcal{L}_K(\mathcal{A})$. In particular, with regard to our goal of comparing asymptotic conditional reasoning to nonmonotonic modal logics, there is no point to consider knowledge-inconsistent modal premises. Moreover, we characterize asymptotic conditional reasoning when the premise is a *flat* modal formula, i.e., a formula without nested occurrences of the modal operator. The flat fragment of a modal logic of knowledge is certainly a very important (if not the most important) syntactically restricted fragment. It consists of the Boolean closure of *knowledge bases*, i.e., of objective theories under the K operator. In particular, the flat fragment in $S5_G$ is extremely powerful and expressive. As shown in (Rosati 1998), via appropriate translations, this fragment captures the well-known formalisms of default logic and logic programming under the stable model semantics. In the present paper we limit our attention to this flat fragment. However, we think that (by slightly more involved methods) our results carry over to the fully nested fragment (i.e., the fragment with nesting depth greater than 1).

Results. By first answering the second of the above-stated research questions, we show the following main result:

Answer to Question 2: φ almost surely entails ψ iff ψ is entailed by φ in $S5_G$.

This result gives a fresh probabilistic semantics to the well-known nonmonotonic modal logic $S5_G$, providing a new justification for $S5_G$ based on probabilistic rationality. At the same time, it provides the answers to our questions 1 and 4, too. In fact, as already mentioned, reasoning in $S5_G$ has precise characterizations in terms of model-theory and complexity, hence the same characterizations now apply to asymptotic conditional reasoning over Kripke models:

Answer to Question 1: φ almost surely entails ψ iff ψ is satisfied by all Kripke models of φ that are universal and have a maximal set of worlds.

Answer to Question 4: Deciding whether φ almost surely entails ψ is Π_2^P -complete.⁴

A theory is *honest* (Halpern and Moses 1985) iff it has exactly one $S5_G$ -model. It has been argued that the epistemic state of a perfectly rational agent is necessarily honest, e.g., it could not be of the form $(KT_1) \vee (KT_2)$ where T_1 and T_2 are theories that contradict each other. Honest premises drastically simplify asymptotic conditional reasoning:

Answer to Question 3: The class of knowledge-consistent premises φ that imply a 0-1 law is exactly the class of honest formulas. In other words, the honest formulas are precisely

⁴ Π_2^P is the complement of $\Sigma_2^P = NP^{NP}$; Θ_2^P is the class of problems solved in PTIME by a logarithmic number of calls to an NP-oracle.

the formulas φ such that, for all ψ , the asymptotic probability that ψ holds in the structures in which φ is valid is either 0 or 1. Moreover, asymptotic conditional reasoning based on honest theories is only Θ_2^P -complete.

Due to space limitations, we can only include the sketches of some results in the present version of the paper.

2 Preliminaries

We assume familiarity with modal logics K and S5. We deal with a propositional alphabet \mathcal{A} such that either \mathcal{A} is finite and fixed, i.e., it is the same for every problem instance, or \mathcal{A} is not bounded, but each problem instance comes along with a finite alphabet \mathcal{A} as part of the input. We denote by $\mathcal{L}(\mathcal{A})$ the set of propositional (or objective) formulas over \mathcal{A} , denote by $\mathcal{L}_K(\mathcal{A})$ the set of modal formulas over \mathcal{A} , and denote by $\mathcal{L}_K^F(\mathcal{A})$ the set of *flat* modal formulas, i.e., the subset of $\mathcal{L}_K(\mathcal{A})$ of formulas over \mathcal{A} satisfying the following abstract syntax:

$$\varphi ::= Kf \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2, \text{ where } f \in \mathcal{L}(\mathcal{A})$$

We also use the symbol **true** to denote the formula $a \vee \neg a$, and the symbol **false** to denote the formula $a \wedge \neg a$.

We now recall some auxiliary definitions that we will use in the following sections (Marek and Truszczyński 1993; Donini *et al.* 1997). Given $\varphi \in \mathcal{L}_K(\mathcal{A})$, we denote by $MA(\varphi)$ the set of *modal atoms* from φ , i.e., the set of subformulas of the form $K\psi$ occurring in φ . In the following, we say that an occurrence of a modal atom $K\psi$ in a formula $\varphi \in \mathcal{L}_K(\mathcal{A})$ is *strict* if it does not lie within the scope of a modal operator.

Given a partition (P, N) of the set $MA(\varphi)$ and a formula $\psi \in \mathcal{L}_K(\mathcal{A})$, we denote by $\psi(P, N)$ the formula obtained from ψ by substituting each strict occurrence in ψ of a modal atom in P with **true**, and each strict occurrence in ψ of a modal atom in N with **false**. Notice that, if $P \cup N$ contains $MA(\psi)$, then $\psi(P, N)$ is a propositional formula.

Let $\varphi \in \mathcal{L}_K(\mathcal{A})$ and let (P, N) be a partition of $MA(\varphi)$. We denote by $obj_\varphi(P, N)$ the propositional formula $obj_\varphi(P, N) = \varphi(P, N) \wedge \bigwedge_{K\psi \in P} \psi(P, N)$.

Given a partition (P, N) of the set $MA(\varphi)$, we say that (P, N) is *S5-consistent with φ* if (P, N) satisfies the following conditions: (1) the propositional formula $obj_\varphi(P, N)$ is satisfiable; (2) for each $K\psi \in N$, the propositional formula $obj_\varphi(P, N) \wedge \neg\psi(P, N)$ is satisfiable. It is immediate to see that there exists a partition of $MA(\varphi)$ S5-consistent with φ iff φ is knowledge-consistent, i.e., there exists an S5-structure \mathcal{S} such that $(w, \mathcal{S}) \models K\varphi$ where w is a world of \mathcal{S} . Finally, given a structure $\mathcal{S} = \langle W, R, V \rangle$ and a world $w \in W$, we say that (P, N) is the partition of $MA(\varphi)$ *satisfied by (w, \mathcal{S})* if, for each $K\psi \in MA(\varphi)$, $(w, \mathcal{S}) \models K\psi$ iff $K\psi \in P$.

3 Strong almost-sure conditional validity

In the work of Halpern and Kapron (Halpern and Kapron 1994), almost sure structure validity is studied by considering all possible Kripke structures equally likely, i.e., uniformly distributed. This amounts to assume that every propositional variable is true with probability 1/2 in a randomly chosen world. Under such an assumption, the

asymptotic probability of φ w.r.t. a propositional alphabet \mathcal{A} corresponds to the limit $\lim_{n \rightarrow \infty} |\mathcal{W}_n^\varphi|/|\mathcal{W}_n|$, where: (i) \mathcal{W}_n denotes the set of all n -structures over \mathcal{A} , i.e., the structures with n worlds of the form $\langle W, R, V \rangle$, where $W = \{1, \dots, n\}$, the accessibility relation R is a binary relation over W , and V is a function mapping each world into a propositional interpretation over \mathcal{A} ; (ii) \mathcal{W}_n^φ denotes the set of all n -structures in which φ holds; (iii) $|S|$ represents the cardinality of a set S . Moreover, under the above uniform probability assumption, asymptotic conditional probability of ψ given φ corresponds to the limit $\lim_{n \rightarrow \infty} |\mathcal{W}_n^{(\varphi \wedge \psi)}|/|\mathcal{W}_n^\varphi|$.

It would be more appealing to consider the notion of *strong almost sure validity*, where the asymptotic probability of a formula ψ is required to be equal to 1 for every possible probability distribution assigning rational truth probabilities to the propositional variables. It turns out that in the context of (Halpern and Kapron 1994) both concepts are equivalent, thus all relevant results of (Halpern and Kapron 1994) extend to strong almost sure validity. In the context of conditional probabilities, these concepts differ, however. In this paper we choose to characterize the notion of *strong almost sure conditional validity*, which is independent of a particular fixed probability distribution. However, we will use (Section 4) the concept of almost sure conditional validity (i.e., the one corresponding to the uniform distribution of Kripke structures) as a tool for establishing (Section 5) our main result on strong almost sure validity.

More formally, we associate to each propositional atom $a \in \mathcal{A}$ a rational probability m such that $0 < m < 1$, which is interpreted as the probability that the proposition a is true. Such an assignment is part of the input. Assignments giving probability 1 (resp. 0) to a are not considered, since in such cases the proposition a is certainly true (resp. false) and all its occurrences in a formula can be eliminated in a simple way. We assume without loss of generality that m is a finite sum of the negative powers of two. Hence, we say that \mathcal{P} is a *probability assignment* over an alphabet \mathcal{A} if \mathcal{P} is a function mapping each propositional symbol from \mathcal{A} to a rational number m such that there exists a finite binary sequence $m_1 \dots m_p$ such that $m = \sum_{i=1}^p \frac{m_i}{2^i}$.

It is immediate to verify the following relationship between the cardinality of a set of interpretations and the number of different n -structures defined over such interpretations.⁵

Lemma 3.1 *Let \mathcal{I} be a set of propositional interpretations, and let \mathcal{W}_n be the set of n -structures defined using the set of interpretations \mathcal{I} . Then, $|\mathcal{W}_n| = |\mathcal{I}|^n \cdot 2^{n^2}$.*

Let $\varphi, \psi \in \mathcal{L}_K(\mathcal{A})$, and let \mathcal{P} be a probability assignment over \mathcal{A} . Then, $p_n^{\mathcal{P}}(\psi|\varphi)$ denotes the probability that ψ is valid in the n -structures in which φ is valid, under the probability assignment \mathcal{P} . We are now ready to define almost-sure and strong almost-sure conditional validity.

Definition 3.2 (\mathcal{P} -almost-sure validity) *Let \mathcal{P} be a probability assignment over \mathcal{A} . We say that ψ conditioned by*

φ is \mathcal{P} -almost-surely valid if $\lim_{n \rightarrow \infty} p_n^{\mathcal{P}}(\psi|\varphi) = 1$, i.e., the asymptotic probability that ψ conditioned by φ is valid is 1 under the probability assignment \mathcal{P} . Conversely, if $\lim_{n \rightarrow \infty} p_n^{\mathcal{P}}(\psi|\varphi) < 1$, then we say that ψ conditioned by φ is not \mathcal{P} -almost-surely valid.

Definition 3.3 (strong-almost-sure validity) *ψ conditioned by φ is strongly almost-surely valid if $\lim_{n \rightarrow \infty} p_n^{\mathcal{P}}(\psi|\varphi) = 1$ for each probability assignment \mathcal{P} over \mathcal{A} . Conversely, if there exists a probability assignment \mathcal{P} over \mathcal{A} such that $\lim_{n \rightarrow \infty} p_n^{\mathcal{P}}(\psi|\varphi) < 1$, then we say that ψ conditioned by φ is not strongly almost-surely valid.*

We denote by $\text{SAS}(\varphi, \mathcal{A})$ the set of formulas ψ from $\mathcal{L}_K(\mathcal{A})$ such that ψ conditioned by φ is strongly almost-surely valid.

4 Counting structures

In this section we prove the correspondence between the notion of stable set in nonmonotonic modal logics and strong almost-sure conditional validity with respect to objective formulas, and establish a first correspondence between S5_G and strong almost-sure conditional validity in modal logic.

From now on, we denote by h the number of propositional interpretations of \mathcal{A} , i.e., $h = 2^{|\mathcal{A}|}$. Moreover, given a formula $f \in \mathcal{L}(\mathcal{A})$, we denote by h_f the number of interpretations of \mathcal{A} satisfying f .

We start our analysis by studying the properties of the set of n -structures for f , i.e., the n -structures in which a propositional formula f is valid. First of all, from Lemma 3.1, it follows that the number of n -structures for f is $h_f \cdot 2^{n^2}$.

Then, we recall the definition of *stable set* of modal formulas. Let f be a satisfiable propositional formula over the propositional alphabet \mathcal{A} . The *stable set* of f in \mathcal{A} (denoted by $\text{Stable}(f, \mathcal{A})$) is the unique set of modal formulas $\mathcal{T} \subset \mathcal{L}_K(\mathcal{A})$ that satisfies the following conditions: (i) for each $\psi \in \mathcal{L}(\mathcal{A})$, $\psi \in \mathcal{T}$ iff $f \supset \psi$ is a tautology; (ii) if $\psi \in \mathcal{T}$ then $K\psi \in \mathcal{T}$; (iii) if $\psi \in \mathcal{L}_K(\mathcal{A}) - \mathcal{T}$ then $\neg K\psi \in \mathcal{T}$; (iv) \mathcal{T} is closed under propositional consequence (Marek and Truszczyński 1993).

Next, we prove the correspondence between formulas strongly almost-surely valid with respect to an objective formula f and the formulas in $\text{Stable}(f, \mathcal{A})$. The proof is easily obtained by extending an analogous result in (Halpern and Kapron 1994).

Lemma 4.1 *Let $f \in \mathcal{L}(\mathcal{A})$ be a satisfiable objective formula. Then, $\text{SAS}(f, \mathcal{A}) = \text{Stable}(f, \mathcal{A})$.*

We now introduce two auxiliary lemmas.

Lemma 4.2 *Let φ be a flat and knowledge-consistent formula, let \mathcal{W}_n^φ be the set of n -structures for φ and let \mathcal{D}_n be the set of n -structures for φ in which all worlds satisfy the same partition (P, N) of $\text{MA}(\varphi)$ S5-consistent with φ . Then, $\lim_{n \rightarrow \infty} |\mathcal{D}_n|/|\mathcal{W}_n^\varphi| = 1$.*

Proof sketch. The proof is divided in two steps: First, we prove that, if φ is a flat and knowledge-consistent formula, $\lim_{n \rightarrow \infty} |\mathcal{D}'_n|/|\mathcal{W}_n^\varphi| = 1$, where \mathcal{D}'_n denotes the set of n -structures for φ in which no world satisfies an S5-inconsistent partition of $\text{MA}(\varphi)$. Then, we concentrate on the set \mathcal{D}'_n : let \mathcal{D}''_n denote the subset of \mathcal{D}'_n in which

⁵We adopt the well-known *random worlds* method (Grove et al. 1996b; 1996a).

two worlds satisfy two different S5-consistent partitions of $MA(\varphi)$ (P_1, N_1), (P_2, N_2), let \mathcal{D}_n^1 be the subset of \mathcal{D}'_n in which all worlds satisfy (P_1, N_1) and let \mathcal{D}_n^2 be the subset of \mathcal{D}'_n in which all worlds satisfy (P_2, N_2). We prove that either $\lim_{n \rightarrow \infty} |\mathcal{D}_n''|/|\mathcal{D}_n^1| = 0$ or $\lim_{n \rightarrow \infty} |\mathcal{D}_n''|/|\mathcal{D}_n^2| = 0$. \square

The following property can be derived by an argument analogous to the proof of Lemma 4.1.

Lemma 4.3 *Let φ be a flat and knowledge-consistent formula and let (P, N) be a partition of $MA(\varphi)$ consistent with φ . Let $\mathcal{W}_n^\varphi(P, N)$ denote the set of n -structures for φ in which all worlds satisfy the partition (P, N) , and let \mathcal{D}_n be the set of n -structures for $obj_\varphi(P, N) \wedge \mu(N)$, where $\mu(N) = \bigwedge_{Kf \in \mathcal{N}} \neg Kf$. Then, $\lim_{n \rightarrow \infty} |\mathcal{D}_n|/|\mathcal{W}_n^\varphi(P, N)| = 1$. Moreover, $\lim_{n \rightarrow \infty} |\mathcal{W}_n^\varphi(P, N)| = (h_{obj_\varphi(P, N)})^n \cdot 2^{n^2}$.*

We now define S5_G-preferred partitions of modal atoms.

Definition 4.4 (S5_G-preferred partition) *Let $\varphi \in \mathcal{L}_K(\mathcal{A})$. A partition (P, N) of $MA(\varphi)$ is S5_G-preferred for φ if (P, N) is S5-consistent with φ and there exists no other partition $(P', N') \neq (P, N)$ of $MA(\varphi)$ such that (i) (P', N') is S5-consistent with φ ; and (ii) the propositional formula $obj_\varphi(P, N) \wedge \neg obj_\varphi(P', N')$ is not satisfiable.*

It is immediate to verify that S5_G-preferred partitions of $MA(\varphi)$ are in one-to-one correspondence with S5_G-expansions of φ : in particular, each such partition (P, N) identifies the S5_G-expansion corresponding to $Stable(obj_\varphi(P, N), \mathcal{A})$.

We are now ready to show a first fundamental step towards the correspondence between S5_G and strong almost-sure conditional validity: For each knowledge-consistent modal formula φ , to compute (strong) almost sure conditional validity we can safely consider only the set of n -structures in which all worlds satisfy one of the partitions of $MA(\varphi)$ that are S5_G-preferred for φ .

Theorem 4.5 *Let φ be a flat, knowledge-consistent formula, let \mathcal{W}_n^φ denote the set of n -structures for φ , and let \mathcal{D}_n be the union of the sets of n -structures for φ in which all worlds satisfy the same partition (P, N) of $MA(\varphi)$, where (P, N) is S5_G-preferred for φ . Then, $\lim_{n \rightarrow \infty} |\mathcal{D}_n|/|\mathcal{W}_n^\varphi| = 1$.*

Proof. Let \mathcal{C}'_n be the union of the sets of n -structures for $obj_\varphi(P, N) \wedge \mu(N)$ for each partition (P, N) of $MA(\varphi)$ that is S5-consistent with φ . First, by Lemma 4.2 and Lemma 4.3 it follows that $\lim_{n \rightarrow \infty} |\mathcal{C}'_n|/|\mathcal{W}_n^\varphi| = 1$. Moreover, let \mathcal{C}_n be the union of the sets of n -structures for $obj_\varphi(P, N) \wedge \mu(N)$ for each partition (P, N) of $MA(\varphi)$ that is S5_G-preferred for φ . By the same lemmas it follows that $\lim_{n \rightarrow \infty} |\mathcal{C}_n|/|\mathcal{D}_n| = 1$. Thus, we have to prove that $\lim_{n \rightarrow \infty} |\mathcal{C}_n|/|\mathcal{C}'_n| = 1$.

Let (P, N) be a partition (P, N) of $MA(\varphi)$ S5_G-preferred for φ , and consider all the partitions (P', N') of $MA(\varphi)$ such that $obj_\varphi(P, N)$ is satisfied by all propositional interpretations satisfying $obj_\varphi(P', N')$. Let $\mathcal{C}_n^{(P, N)}$ be the set of n -structures for $obj_\varphi(P, N) \wedge \mu(N)$, and let $\mathcal{C}_n^{(P', N')}$ be the

union of the set of n -structures for $obj_\varphi(P', N') \wedge \mu(N')$ for each such partition (P', N') . Obviously, $\mathcal{C}_n^{(P, N)} \subseteq \mathcal{C}_n^{(P', N')}$ (since (P, N) is one of such partitions (P', N')).

We now prove that $\lim_{n \rightarrow \infty} \frac{|\mathcal{C}_n^{(P, N)}|}{|\mathcal{C}_n^{(P', N')}|} = 1$. Let $\mathcal{C}_n'' = \mathcal{C}_n^{(P', N')} - \mathcal{C}_n^{(P, N)}$. That is, \mathcal{C}_n'' is the set of n -structures for $obj_\varphi(P', N') \wedge \mu(N')$ for each partition (P', N') that is S5-consistent with φ and such that $obj_\varphi(P, N)$ is satisfied by all propositional interpretations satisfying $obj_\varphi(P', N')$, and there exists at least an interpretation satisfying $obj_\varphi(P, N) \wedge \neg obj_\varphi(P', N')$. Now let k be the number of interpretations of \mathcal{A} satisfying $obj_\varphi(P, N)$: It is immediate to verify that there can be at most k such different partitions (P', N') . Moreover, for each such partition (P', N') , there exists at least a propositional interpretation \mathcal{I} that satisfies $obj_\varphi(P, N)$ and does not satisfy $obj_\varphi(P', N')$. Therefore, from Lemma 4.3 we have that $\frac{|\mathcal{C}_n''|}{|\mathcal{C}_n^{(P', N')}|} \leq \frac{k \cdot (k-1)^n \cdot 2^{n^2}}{k^n \cdot 2^{n^2}} = k \cdot ((k-1)/k)^n$. Consequently, $\lim_{n \rightarrow \infty} \frac{|\mathcal{C}_n''|}{|\mathcal{C}_n^{(P', N')}|} = 0$, which proves that $\lim_{n \rightarrow \infty} \frac{|\mathcal{C}_n^{(P, N)}|}{|\mathcal{C}_n^{(P', N')}|} = 1$. Hence, $\lim_{n \rightarrow \infty} \frac{|\mathcal{C}_n|}{|\mathcal{C}'_n|} = 1$. \square

5 Asymptotic conditional probability and S5_G

In this section we prove the main result of the paper, which establishes the correspondence between strong almost-sure conditioned validity and entailment in the logic S5_G. To this aim, we need some preliminary definitions and properties.

Let \mathcal{A}' be the set of propositional atoms a^i such that $a \in \mathcal{A}$ and $1 \leq i \leq m$. We define the *canonical probability assignment* \mathcal{P}_c over \mathcal{A}' as follows: $\mathcal{P}_c(a) = \frac{1}{2}$ for each $a \in \mathcal{A}'$. Let \mathcal{A} be a propositional alphabet and let \mathcal{I}_f be the set of propositional interpretations over \mathcal{A} that satisfy f . The *canonical probability* of a propositional formula f , denoted by $cp(f)$, is defined as $cp(f) = \frac{|\mathcal{I}_f|}{2^{|\mathcal{A}|}}$.

The following auxiliary lemma establishes a sufficient condition over the partitions of $MA(\varphi)$ which implies that $\lim_{n \rightarrow \infty} p_n^{\mathcal{P}_c}(\psi|\varphi) < 1$.

Lemma 5.1 *Let $\varphi \in \mathcal{L}_K^F(\mathcal{A})$. If there exists a partition (P, N) of $MA(\varphi)$ such that: (1) (P, N) is S5_G-preferred for φ ; (2) for each partition (P', N') that is S5_G-preferred for φ , $cp(obj_\varphi(P, N)) \geq cp(obj_\varphi(P', N'))$; (3) ψ does not hold in the structures for $obj_\varphi(P, N)$, then $\lim_{n \rightarrow \infty} p_n^{\mathcal{P}_c}(\psi|\varphi) < 1$.*

Let $a \in \mathcal{A}$ and let $\mathcal{P}(a) = k$ where k is a rational number satisfying the definition of probability assignment. Then, we can construct a propositional formula $f^k(a)$ defined over an alphabet $\mathcal{A}^k = \{a^1, \dots, a^{m_k}\}$ (where m_k is a number depending on k) such that $\mathcal{P}(a) = cp(f^k(a))$. Then, given a probability assignment \mathcal{P} over \mathcal{A} , we define $\tau_{\mathcal{P}}(\varphi)$ as the formula obtained from φ by replacing, for each $a \in \mathcal{A}$, each occurrence of a in φ with $f^{\mathcal{P}(a)}(a)$. Now, in order to prove our main result, we need some auxiliary lemmas.

Lemma 5.2 Let $\psi \in \mathcal{L}_K(\mathcal{A})$, and let $f \in \mathcal{L}(\mathcal{A})$. Then, $\psi \in \text{SAS}(f, \mathcal{A})$ iff $\tau_{\mathcal{P}}(\psi) \in \text{SAS}(\tau_{\mathcal{P}}(f), \mathcal{A}^k)$.

Lemma 5.3 Let $\varphi, \psi \in \mathcal{L}_K(\mathcal{A})$. then, for each partition (P, N) that is S5_G -preferred for φ , the partition $(\tau_{\mathcal{P}}(P), \tau_{\mathcal{P}}(N))$ is S5_G -preferred for $\tau_{\mathcal{P}}(\varphi)$, while for each partition $(\tau_{\mathcal{P}}(P), \tau_{\mathcal{P}}(N))$ that is S5_G -preferred for $\tau_{\mathcal{P}}(\varphi)$, the partition (P, N) is S5_G -preferred for φ .

Lemma 5.4 Let $\varphi \in \mathcal{L}_K(\mathcal{A})$. If a partition (P, N) of $\text{MA}(\varphi)$ is S5_G -preferred for φ , then the stable set of $\text{obj}_{\varphi}(P, N)$ in \mathcal{A} is an S5_G -expansion for φ . Moreover, if a set $\mathcal{T} \subset \mathcal{L}_K(\mathcal{A})$ is an S5_G -expansion for φ , then there exists a partition (P, N) of $\text{MA}(\varphi)$ such that (P, N) is S5_G -preferred for φ and $\text{Stable}(\text{obj}_{\varphi}(P, N), \mathcal{A}) = \mathcal{T}$.

Theorem 5.5 Let $\varphi \in \mathcal{L}_K^F(\mathcal{A})$, $\psi \in \mathcal{L}_K(\mathcal{A})$, and let φ be a knowledge-consistent formula. Then, ψ conditioned by φ is strongly almost-surely valid iff $\varphi \models_{\text{S5}_G} \psi$.

Proof. First, if $\varphi \models_{\text{S5}_G} \psi$, then, from definition of entailment in S5_G and from Lemma 5.4 it follows that, for each partition (P, N) that is S5_G -preferred for φ , ψ belongs to $\text{Stable}(\text{obj}_{\varphi}(P, N), \mathcal{A})$, consequently, by Lemma 4.1, $\psi \in \text{SAS}(\text{obj}_{\varphi}(P, N), \mathcal{A})$. Hence, by Lemma 5.2 and Lemma 5.3, for each probability assignment \mathcal{P} over \mathcal{A} , $\tau_{\mathcal{P}}(\psi) \in \text{SAS}(\tau_{\mathcal{P}}(\text{obj}_{\varphi}(P, N)), \mathcal{A}^k)$. Therefore, by Theorem 4.5, $\lim_{n \rightarrow \infty} p_n^{\mathcal{P}}(\tau_{\mathcal{P}}(\psi) | \tau_{\mathcal{P}}(\varphi)) = 1$, which implies that ψ conditioned by φ is strongly almost-surely valid.

Conversely, if $\varphi \not\models_{\text{S5}_G} \psi$, then by Lemma 5.4 it follows that there exists a partition (P, N) that is S5_G -preferred for φ and such that ψ does not belong to $\text{Stable}(\text{obj}_{\varphi}(P, N), \mathcal{A})$, consequently, by Lemma 4.1, $\psi \notin \text{SAS}(\text{obj}_{\varphi}(P, N), \mathcal{A})$. Therefore, by Lemma 5.2, $\tau_{\mathcal{P}}(\psi) \notin \text{SAS}(\tau_{\mathcal{P}}(\text{obj}_{\varphi}(P, N)), \mathcal{A}^k)$. Moreover, it is immediate to verify the existence a probability assignment \mathcal{P}' over \mathcal{A} such that, for each partition (P', N') that is S5_G -preferred for φ , $\text{cp}(\tau_{\mathcal{P}'}(\text{obj}_{\varphi}(P, N))) \geq \text{cp}(\tau_{\mathcal{P}'}(\text{obj}_{\varphi}(P', N')))$. Consequently, by Lemma 5.1, it follows that $\lim_{n \rightarrow \infty} p_n^{\mathcal{P}'}(\tau_{\mathcal{P}'}(\psi) | \tau_{\mathcal{P}'}(\varphi)) < 1$, i.e., ψ conditioned by φ is not strongly almost-surely valid. \square

6 Complexity results

As explained above, we consider both the case in which \mathcal{A} is a fixed finite alphabet and the case in which it is considered as finite but not fixed in advance. If \mathcal{A} is finite and fixed, then, based on the algorithm for entailment in S5_G reported in (Donini *et al.* 1997) it can be easily shown that entailment in S5_G can be decided in polynomial time, and therefore, by Theorem 5.5, strong almost-sure conditional validity can be decided in polynomial time as well. Consider now the case of a finite, non-fixed alphabet \mathcal{A} .

Theorem 6.1 Let $\varphi \in \mathcal{L}_K^F(\mathcal{A})$, $\psi \in \mathcal{L}_K(\mathcal{A})$, and let φ be a knowledge-consistent formula. Deciding whether ψ conditioned by φ is strongly almost-surely valid is a Π_2^P -complete problem.

Proof. Follows immediately from Theorem 5.5 and from the fact that the entailment problem $\varphi \models_{\text{S5}_G} \psi$ when $\varphi \in$

$\mathcal{L}_K^F(\mathcal{A})$ is Π_2^P -complete, which is an immediate consequence of Corollary 4.12 of (Donini *et al.* 1997) and of Theorem 11 of (Rosati 1998). \square

Next, we study a subclass of flat modal formulas for which deciding strong almost-sure conditional validity is computationally easier than in the general case: the class of *honest* formulas, introduced by (Halpern and Moses 1985). A flat formula φ is honest iff it has exactly one S5_G -model. It is known that deciding whether φ is honest is a Θ_2^P -complete problem. Moreover, if φ is honest, then deciding the entailment $\varphi \models_{\text{S5}_G} \psi$ is also Θ_2^P -complete. Therefore, from Theorem 5.5 the following property holds.

Theorem 6.2 Let $\varphi \in \mathcal{L}_K^F(\mathcal{A})$, $\psi \in \mathcal{L}_K(\mathcal{A})$, and let φ be a honest formula. Deciding whether ψ conditioned by φ is strongly almost-surely valid is Θ_2^P -complete.

Furthermore, the notion of honesty precisely characterizes the subclass of formulas, among the formulas φ such that $K\varphi$ is S5 -consistent, that condition according to a 0-1 law, i.e., such that either $\lim_{n \rightarrow \infty} p_n^{\mathcal{P}}(\psi | \varphi) = 1$ for each probability assignment \mathcal{P} or $\lim_{n \rightarrow \infty} p_n^{\mathcal{P}}(\psi | \varphi) = 0$ for each probability assignment \mathcal{P} , which is immediately implied by Theorem 5.5 and by the fact that a honest formula has a single S5_G -model.

7 Conclusions

The present work can be extended in several directions. In particular, we are currently investigating the following issues: (i) strong almost-sure validity for (classes of) conditioning formulas with nested occurrences of modalities; (ii) extending the study of asymptotic conditional validity to the framework of multimodal logic; (iii) computing \mathcal{P} -almost-sure conditional validity, i.e., the value of the asymptotic conditional validity for a given probability distribution \mathcal{P} of the truth of primitive propositions.

References

- Jean-Marie Le Bars. Fragments of existential second-order logic without 0-1 laws. In *Proc. of LICS'98*, pages 525–536, 1998.
- Jean-Marie Le Bars. The 0-1 law fails for frame satisfiability of propositional modal logic. In *Proc. of LICS 2002*, pages 225–234, 2002.
- R. Carnap. *Meaning and necessity*. University of Chicago Press, 1947.
- K. Compton. 0-1 laws in logic and combinatorics. In *Proc. 1987 Nato Adv. Study Inst. on Algorithms and Order*, pages 353–383. Reidel, Dordrecht, Netherlands, 1988.
- Francesco M. Donini, Daniele Nardi, and Riccardo Rosati. Ground nonmonotonic modal logics. *J. of Log. and Comp.*, 7(4):523–548, August 1997.
- R. Fagin. Probabilities on finite models. *J. of Symbolic Logic*, 41:50–58, 1976.
- Y.V. Glebskii, D.I. Kogan, M.I. Liogonki, and V.A. Talanov. Range and degree of realizability of formulas in the restricted predicate calculus. *Cybernetics*, 5:142–154, 1969.

Georg Gottlob. Review of a Carnapian extension of S5. In Ewa Orłowska, editor, *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa*. Physica Verlag (Springer), 1999.

Adam Grove, Joseph Halpern, and Daphne Koller. Asymptotic conditional probabilities: The non-unary case. *J. of Symbolic Logic*, 61(1):250–276, 1996.

Adam Grove, Joseph Halpern, and Daphne Koller. Asymptotic conditional probabilities: The unary case. *SIAM J. on Computing*, 25(1):1–51, 1996.

Josef Y. Halpern and Bruce Kapron. Zero-one laws for modal logic. *Annals of Pure and Applied Logics*, 69:157–193, 1994.

Josef Y. Halpern and Yoram Moses. Towards a theory of knowledge and ignorance: Preliminary report. In K. Apt, editor, *Logic and models of concurrent systems*. Springer, 1985.

Phokion G. Kolaitis and Moshe Y. Vardi. 0-1 laws and decision problems for fragments of second-order logic. *Information and Computation*, 87(1/2):301–337, 1990.

Vladimir Lifschitz. Minimal belief and negation as failure. *Artificial Intelligence*, 70:53–72, 1994.

W. Marek and M. Truszczyński. *Nonmonotonic Logics: Context-Dependent Reasoning*. Springer, 1993.

R. C. Moore. Semantical considerations on nonmonotonic logic. *Artificial Intelligence*, 25:75–94, 1985.

Riccardo Rosati. Embedding negation as failure into minimal knowledge. In *Proc. of ECAI'98*, pages 28–32. John Wiley & Sons, 1998.

Y. Shoham. Nonmonotonic logics: Meaning and utility. In *Proc. of IJCAI'87*, pages 388–392, 1987.

Robert Stalnaker. A note on non-monotonic modal logic. *Artificial Intelligence*, 64(2):183–196, 1993.